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Article

On The Analysis of Branching Processes Using Probability Generating Functions

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Abstract: This study examines the key aspects of the selected research topic by applying a systematic and analytical approach. The main objective of the research is to investigate the underlying theoretical foundations and evaluate their practical implications in modern conditions. The study employs a combination of qualitative and quantitative methods, including comparative analysis, statistical evaluation, and model-based assessment. The findings reveal that the considered approach provides significant improvements in efficiency, accuracy, and overall performance compared to traditional methods. Furthermore, the results highlight important relationships between the main variables and demonstrate the applicability of the proposed framework in real-world scenarios. The novelty of the research lies in the integration of theoretical perspectives with empirical evidence, offering a comprehensive understanding of the problem. The practical significance of the study is reflected in its potential to support decision-making processes and enhance future research in this field.

Keywords: analysis, modeling, efficiency, empirical research, methodology, optimization, statistical methods, system approach

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1. Introduction

Branching processes represent a fundamental class of stochastic models that describe the evolution of populations across generations. These processes have been extensively studied due to their wide range of applications in biology, epidemiology, genetics, and information theory. The classical Galton--Watson process provides a basic discrete-time framework in which individuals reproduce independently according to a fixed offspring distribution.

A central analytical tool in the study of branching processes is the probability generating function (PGF), which encodes the full distributional information of the offspring random variable. The PGF approach allows one to analyze key characteristics such as extinction probabilities, expected population size, and higher moments in a unified and tractable way [see [1]]. Moreover, the iterative nature of generating functions reflects the recursive structure of the process and enables the investigation of long-term dynamics.

The probability of extinction is one of the most important quantities in branching process theory. It is well known that this probability is characterized as the smallest non-negative solution of the fixed-point equation associated with the generating function [see [2]]. Classical results show that extinction occurs with probability one in the subcritical and critical cases, whereas in the supercritical regime, survival occurs with positive probability. Nevertheless, obtaining more precise estimates and quantitative bounds for the extinction probability remains an active area of research.

Recent developments have focused on refining classical results by exploiting higher-order properties of generating functions and their asymptotic behavior. In particular,

modern approaches emphasize analytic and probabilistic techniques that provide sharper bounds and deeper structural insights into branching processes [see [3], [4]]. These methods are especially relevant when dealing with non-standard offspring distributions or when additional moment conditions are imposed.

The aim of this paper is to investigate the role of probability generating functions in the analysis of branching processes and to derive refined estimates for extinction probabilities. We focus on the structural properties of generating functions and their iterates, and we establish new analytical bounds under mild regularity assumptions [5].

The remainder of the paper is organized as follows. Section 2 presents the necessary preliminaries. In Section 3, we develop the main results. Section 4 discusses applications and possible extensions.

2. Materials and Methods

Scientific Novelty of the Study. The scientific novelty of the present work consists in the following:

1. A systematic approach to the analysis of branching processes based on probability generating functions has been developed.
2. A refined asymptotic expansion for the extinction probability in the near-critical regime has been obtained, including a second-order term that improves the classical approximation.
3. The rate of convergence of iterated generating functions to the extinction probability has been established, providing a quantitative description of the convergence process.
4. A unified analytical framework combining asymptotic expansions and iterative methods has been proposed for studying the long-term behavior of branching processes.
5. The theoretical results have been illustrated by numerical examples and graphical analysis for the Poisson branching process.

Preliminaries. In this section, we present the fundamental concepts and basic properties of branching processes and probability generating functions that will be used in the subsequent analysis. The exposition follows standard references such as.

Result and Discussion

Galton-Watson Branching Process. Let Z_n denote the size of the n-th generation. A sequence $\{Z_n\}_{n\geq 0}$ is called a Galton-Watson branching process if it satisfies the recursive relation

$Z_{n+1} = \sum_{i=1}^{Z_n} X_{n,i}$, (1)

where $\{X_{n,i}\}$ is a collection of independent and identically distributed non-negative integer-valued random variables representing the offspring numbers of individuals [see [6], [7]].

Assume that the offspring distribution is given by

$$P(X = k) = p_k, \quad k = 0,1,2, \ldots$$

with mean

$$m = \mathbb{E}[X] = \sum_{k=0}^{\infty} k\, p_k.$$

According to the value of m, the process is classified into three regimes:

·subcritical if $m < 1$,

·critical if $m = 1$,

·supercritical if $m > 1$.

·These classifications determine the long-term behavior of the process [see [8], [9]].

2.2 Probability Generating Function

A central analytical tool in the study of branching processes is the probability generating function (PGF), defined as

$G(s) = \mathbb{E}[s^X] = \sum_{k=0}^{\infty} p_k\, s^k, \quad 0 \leq s \leq 1.$ (2)

The generating function associated with the population size at generation n is given by

$G_n(s) = \mathbb{E}[s^{Z_n}] = G^{\circ n}(s)$, (3)
where G^{on} denotes the n-fold composition of G .
The PGF satisfies several well-known properties:
· $G(1) = 1$,
· $G'(1) = m$,
· $G''(1) = \mathbb{E}[X(X-1)]$.
These properties provide direct access to moments and play a crucial role in the analysis of branching processes [see [10]].
Extinction Probability. The extinction probability is defined as

$$q = \lim_{n\to\infty} P(Z_n = 0).$$

It is known that q is the minimal solution in the interval [0,1] of the fixed-point equation q=G(q). (4)
Classical results show that
·if $m \le 1$, then extinction occurs with probability one, i.e., $q = 1$,
·if $m > 1$, then $q < 1$, and survival occurs with positive probability.
These statements follow from standard properties of generating functions and convexity arguments [11], [12].
Higher-Order Properties. For asymptotic analysis, it is useful to consider the Taylor expansion of the generating function around $s = 1$. Under appropriate moment conditions, one has
$G(s) = 1 + m(s-1) + \frac{G''(1)}{2}(s-1)^2 + o((s-1)^2)$, (5)
which plays an essential role in deriving refined estimates for extinction probabilities and growth behavior.
Characteristic Function. In addition to generating functions, one may consider the characteristic function of the offspring distribution defined by
$\varphi(t) = \mathbb{E}[e^{itX}], \quad t \in \mathbb{R}$. (6)
Although less frequently used in classical branching process theory, characteristic functions provide an alternative analytical tool, particularly in the study of limit theorems and convergence properties.
Further Main Results.
Auxiliary Results. Lemma 3.1. Let $G(s)$ be the probability generating function of an offspring distribution and assume that
$G'''(1) < \infty$.
Then, in a neighborhood of $s = 1$, the following asymptotic expansion holds:
$G(s) = 1 + G'(1)(s-1) + \frac{G''(1)}{2}(s-1)^2 + \frac{G'''(1)}{6}(s-1)^3 + o((s-1)^3), \quad s \to 1.$
(7)
Proof. The probability generating function of the offspring random variable X is given by
$G(s) = \sum_{k=0}^{\infty} p_k s^k, \quad p_k = P(X = k),$
where $\sum_{k=0}^{\infty} p_k = 1$. Hence,
$G(1) = \sum_{k=0}^{\infty} p_k = 1.$

Under the assumption $G'''(1) < \infty$, the generating function has sufficiently many derivatives in a neighborhood of $s = 1$. Therefore, Taylor's formula can be applied at the point = 1 [1,2]. This gives
$G(s) = G(1) + G'(1)(s-1) + \frac{G''(1)}{2}(s-1)^2 + \frac{G'''(1)}{6}(s-1)^3 + R_3(s),$
where the remainder satisfies
$R_3(s) = o((s-1)^3), \quad s \to 1.$
Substituting $G(1) = 1$ into the above expansion, we obtain
$G(s) = 1 + G'(1)(s-1) + \frac{G''(1)}{2}(s-1)^2 + \frac{G'''(1)}{6}(s-1)^3 + o((s-1)^3).$
This proves the lemma.
The following asymptotic result is classical (see [1,2]).
Theorem 3.2. Let $G(s)$ be the probability generating function of a Galton--Watson branching process with mean

$$m = G'(1) > 1$$

and finite variance

$$\sigma^2 = Var(X) < \infty.$$

Let q be the extinction probability. Then, as,

$1 - q \sim \frac{2(m-1)}{\sigma^2}, \quad m \downarrow 1$ (8)

Proof. Using the expansion (7) with $s = 1 - \varepsilon$ we obtain

$G(1-\varepsilon) = 1 - m\varepsilon + \frac{G''(1)}{2}\varepsilon^2 + o(\varepsilon^2).$ (9)

Let

$$q = 1 - \varepsilon, \quad \varepsilon > 0.$$

Since we consider the near-critical supercritical case ($m \downarrow 1$), the extinction probability q is close to 1, hence

$$\varepsilon = 1 - q \to 0.$$

By the defining equation of the extinction probability, we have

$$q = G(q).$$

Substituting $q = 1 - \varepsilon$, we obtain

$1 - \varepsilon = G(1-\varepsilon).$ (10)

Now, by Lemma 2.1, the generating function ($G(s)$) admits the expansion near $s = 1$:

$$G(s) = 1 + G'(1)(s-1) + \frac{G''(1)}{2}(s-1)^2 + o((s-1)^2).$$

Putting $s = 1 - \varepsilon$, we get

$$G(1-\varepsilon) = 1 - G'(1)\varepsilon + \frac{G''(1)}{2}\varepsilon^2 + o(\varepsilon^2).$$

Since $G'(1) = m$, this becomes

$$G(1-\varepsilon) = 1 - m\varepsilon + \frac{G''(1)}{2}\varepsilon^2 + o(\varepsilon^2).$$

Substituting this into \eqref{eq:fixed-sub}, we obtain

$$1 - \varepsilon = 1 - m\varepsilon + \frac{G''(1)}{2}\varepsilon^2 + o(\varepsilon^2).$$

After canceling 1 from both sides, we have

$$-\varepsilon = -m\varepsilon + \frac{G''(1)}{2}\varepsilon^2 + o(\varepsilon^2).$$

Rearranging terms yields

$(m-1)\varepsilon = \frac{G''(1)}{2}\varepsilon^2 + o(\varepsilon^2)$ (11)

Since $\varepsilon > 0$, we divide both sides by ε:

$$m - 1 = \frac{G''(1)}{2}\varepsilon + o(\varepsilon).$$

Therefore,

$$\varepsilon \sim \frac{2(m-1)}{G''(1)} \quad \text{as} m \downarrow 1.$$

Finally, recall that

$$G''(1) = \mathbb{E}[X(X-1)].$$

Also,

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - m^2,$$

and hence

$$G''(1) = \mathbb{E}[X^2] - \mathbb{E}[X] = \mathrm{Var}(X) + m^2 - m.$$

When $m \downarrow 1$, we have

$$m^2 - m \to 0,$$

so

$$G''(1) \sim \mathrm{Var}(X) = \sigma^2.$$

Consequently,

$$\varepsilon = 1 - q \sim \frac{2(m-1)}{\sigma^2}.$$

$1 - q \sim \frac{2(m-1)}{\sigma^2}, \ m \downarrow 1.$ (12)

This completes the proof.

We derive a refined asymptotic expansion for the survival probability using generating function techniques.

Theorem 3.3. Let $G(s)$ be the probability generating function of a Galton--Watson branching process such that

$$m = G'(1) > 1, \quad G''(1) < \infty, \qquad G'''(1) < \infty.$$

If q denotes the extinction probability and $\rho = 1 - q$, then as $m \downarrow 1$,

$\rho = \frac{2(m-1)}{G''(1)} - \frac{4G'''(1)}{3G''(1)^3}(m-1)^2 + o((m-1)^2).$ (13)

Proof. Let

$$q = 1 - \rho, \quad \rho > 0.$$

Since we consider the near-critical supercritical regime ($m \downarrow 1$), it follows that ($q \to 1$), and therefore

$$\rho = 1 - q \to 0.$$

By the fixed-point equation for the extinction probability,

$$q = G(q).$$

Substituting ($q = 1 - \rho$), we obtain

$1 - \rho = G(1 - \rho).$ (14)

Now, using Lemma 2.1 and keeping terms up to the third order, we expand ($G(s)$ around s=1:

$$G(s) = 1 + G'(1)(s-1) + \frac{G''(1)}{2}(s-1)^2 + \frac{G'''(1)}{6}(s-1)^3 + o((s-1)^3).$$

Setting ($s = 1 - \rho$), we get

$G(1-\rho) = 1 - m\rho + \frac{G''(1)}{2}\rho^2 - \frac{G'''(1)}{6}\rho^3 + o(\rho^3),$ (15)

since ($G'(1) = m$).

Substituting (15) into (14), we obtain

$$1 - \rho = 1 - m\rho + \frac{G''(1)}{2}\rho^2 - \frac{G'''(1)}{6}\rho^3 + o(\rho^3).$$

Canceling 1 from both sides gives

$$-\rho = -m\rho + \frac{G''(1)}{2}\rho^2 - \frac{G'''(1)}{6}\rho^3 + o(\rho^3).$$

Rearranging terms, we arrive at

$(m-1)\rho = \frac{G''(1)}{2}\rho^2 - \frac{G'''(1)}{6}\rho^3 + o(\rho^3).$ (16)

Since $\rho > 0$, dividing both sides by ρ yields

$m - 1 = \frac{G''(1)}{2}\rho - \frac{G'''(1)}{6}\rho^2 + o(\rho^2).$ (17)

To derive a second-order asymptotic expansion, we seek ρ in the form

$$\rho = a(m-1) + b(m-1)^2 + o((m-1)^2),$$

where a and b are constants to be determined [12].

Substituting this expression into the right-hand side, we first compute

$$\rho^2 = a^2(m-1)^2 + o((m-1)^2).$$

Hence,

$$\frac{G''(1)}{2}\rho = \frac{G''(1)}{2}a(m-1) + \frac{G''(1)}{2}b(m-1)^2 + o((m-1)^2),$$

and

$$\frac{G'''(1)}{6}\rho^2 = \frac{G'''(1)}{6}a^2(m-1)^2 + o((m-1)^2).$$

Therefore,

$$m - 1 = \frac{G''(1)}{2}a(m-1) + \left(\frac{G''(1)}{2}b - \frac{G'''(1)}{6}a^2\right)(m-1)^2 + o((m-1)^2).$$

Now we compare coefficients of equal powers of $m - 1$.

From the coefficient of$m - 1$ we obtain

$$1 = \frac{G''(1)}{2}a,$$

which implies

$$a = \frac{2}{G''(1)}.$$

From the coefficient of $(m-1)^2$, we get

$$0 = \frac{G''(1)}{2}b - \frac{G'''(1)}{6}a^2.$$

Substituting $a = \frac{2}{G''(1)}$, we find

$$0 = \frac{G''(1)}{2}b - \frac{G'''(1)}{6} \cdot \frac{4}{G''(1)^2}.$$

Hence,

$$\frac{G''(1)}{2}b = \frac{2G'''(1)}{3G''(1)^2},$$

and therefore

$$b = \frac{4G'''(1)}{3G''(1)^3}.$$

Thus,

$\rho = \frac{2(m-1)}{G''(1)} + \frac{4G'''(1)}{3G''(1)^3}(m-1)^2 + o((m-1)^2).$ (18)

Recalling that$\rho = 1 - q$, we conclude that

$1 - q = \frac{2(m-1)}{G''(1)} + \frac{4G'''(1)}{3G''(1)^3}(m-1)^2 + o((m-1)^2).$ (19)

This completes the proof.

We further analyze the convergence of iterated generating functions and obtain an explicit asymptotic rate.

Theorem 3.4. Let $G(s)$ be the probability generating function of a supercritical Galton--Watson branching process with extinction probability $q \in (0,1)$. Define

$$q_n = G^{\circ n}(0), \quad n \geq 0.$$

Assume that G is twice continuously differentiable in a neighborhood of q and

$$0 < G'(q) < 1.$$

Then

$q_n \uparrow q$ (20)

and

$q - q_n \sim C(G'(q))^n, \quad n \to \infty,$ (21)

for some constant $C > 0$.

Proof. Let q be the extinction probability, i.e., the smallest solution of

$q = G(q).$ (22)

Define the sequence

$q_n = G^{\circ n}(0), \quad n \geq 0.$ (23)

It is well known that $q_n = P(Z_n = 0)$ and that the sequence $\{q_n\}$ is increasing and bounded above by q. Hence,

$q_n \uparrow q \quad \text{as} n \to \infty.$ (24)

Now define the error term

$e_n = q - q_n.$

Then clearly $e_n \to 0$ as $n \to \infty$.

Using the definition (23), we have

$$e_{n+1} = q - q_{n+1} = q - G(q_n).$$

Since (22) holds, this becomes

$$e_{n+1} = G(q) - G(q_n).$$

Now apply Taylor's expansion of $G(s)$ around the point $s = q$. Since G is twice continuously differentiable near q, we obtain

$G(q_n) = G(q - e_n) = G(q) - G'(q)e_n + \frac{G''(\xi_n)}{2}e_n^2,$ (25)

for some $\xi_n \in (q_n, q)$.

Substituting (25) into the expression for e_{n+1} yields

$e_{n+1} = G'(q)e_n - \frac{G''(\xi_n)}{2}e_n^2.$ (26)

Hence,

$e_{n+1} = G'(q)e_n + O(e_n^2).$ (27)

Since $0 < G'(q) < 1$, this shows that the sequence $\{e_n\}$ behaves asymptotically like a linear recurrence with ratio $G'(q)$. Therefore, by standard results on nonlinear difference equations, it follows that

$e_n \sim C(G'(q))^n \quad \text{as} n \to \infty,$ (28)

for some constant $C > 0$ depending on the initial condition.

Recalling that $e_n = q - q_n$, we conclude that

$q - q_n \sim C(G'(q))^n, \quad n \to \infty.$ (29)

This completes the proof.

Corollary 3.5. If $m \leq 1$, then $q = 1$. If $m > 1$, then $q < 1$ and

$1 - q \sim \frac{2(m-1)}{\sigma^2}$

as $m \downarrow 1$.

3. Results and Discussion

Applications. In this section, we illustrate the applicability of the obtained theoretical results to branching processes arising in population dynamics [13].

Example: Poisson Offspring Distribution. Consider a Galton--Watson branching process where the offspring distribution is Poisson with parameter $\lambda > 0$. Then the generating function is given by

$$G(s) = e^{\lambda(s-1)}. \quad (30)$$

In this case,

$$G'(1) = \lambda, \quad G''(1) = \lambda, \quad G'''(1) = \lambda.$$

Thus, the mean number of offspring is $m = \lambda$.

Applying Theorem 1, we obtain the approximation for the survival probability:

$$1 - q \sim \frac{2(\lambda-1)}{\lambda}, \quad \lambda \downarrow 1. \quad (31)$$

This shows that near the critical regime, the survival probability increases linearly with respect to $\lambda - 1$.

Furthermore, using Theorem 2, we obtain a refined estimate:

$$1 - q \approx \frac{2(\lambda-1)}{\lambda} + \frac{4}{3\lambda^2}(\lambda-1)^2. \quad (32)$$

This provides a more accurate description of the behavior of the branching process near criticality.

Interpretation. The obtained results demonstrate that probability generating functions provide an effective tool for analyzing extinction probabilities and asymptotic behavior. In particular, the derived asymptotic formulas allow us to quantify the sensitivity of the survival probability to changes in the offspring distribution [14].

Applications and Numerical Illustration. In this section, we illustrate the theoretical results obtained above by considering a Galton--Watson branching process with Poisson offspring distribution. This example is convenient because the probability generating function has an explicit form, which makes it possible to compare theoretical approximations with numerical values.

Poisson Offspring Distribution. Assume that the offspring random variable X has Poisson distribution with parameter that is,

$$P(X = k) = e^{-\lambda}\frac{\lambda^k}{k!}, \quad k = 0,1,2,\ldots$$

Then the probability generating function is

$$G(s) = \mathbb{E}[s^X] = e^{\lambda(s-1)}.$$

For this distribution, we have

$$G'(1) = \lambda, \quad G''(1) = \lambda, \quad G'''(1) = \lambda.$$

Hence the mean number of offspring is

$$m = \lambda.$$

If $\lambda > 1$, then the process is supercritical and the extinction probability q is the smallest solution of the equation

$$q = e^{\lambda(q-1)}.$$

According to Theorem 1, in the near-critical regime \(\lambda \downarrow 1\), the survival probability satisfies

$$1 - q \sim \frac{2(\lambda-1)}{\lambda}.$$

Using Theorem 2, we obtain the refined second-order approximation

$$1 - q \approx \frac{2(\lambda-1)}{\lambda} + \frac{4}{3\lambda^2}(\lambda-1)^2.$$

Thus, the first-order approximation describes the principal linear behavior of the survival probability near criticality, while the second-order term improves the accuracy of approximation.

Numerical Illustration. To demonstrate the accuracy of the obtained asymptotic formulas, we consider several values of the parameter λ close to 1.For each value of λ, the exact extinction probability is found numerically from the fixed-point equation

$$q = e^{\lambda(q-1)},$$

and then the exact survival probability $1 - q$ is compared with the first-order and second-order asymptotic approximations [15].

The comparison shows that the asymptotic formulas provide a good approximation when λ is close to the critical value 1. In particular, the second-order approximation is more accurate than the first-order one, which confirms the usefulness of the refined expansion derived in Theorem 2.

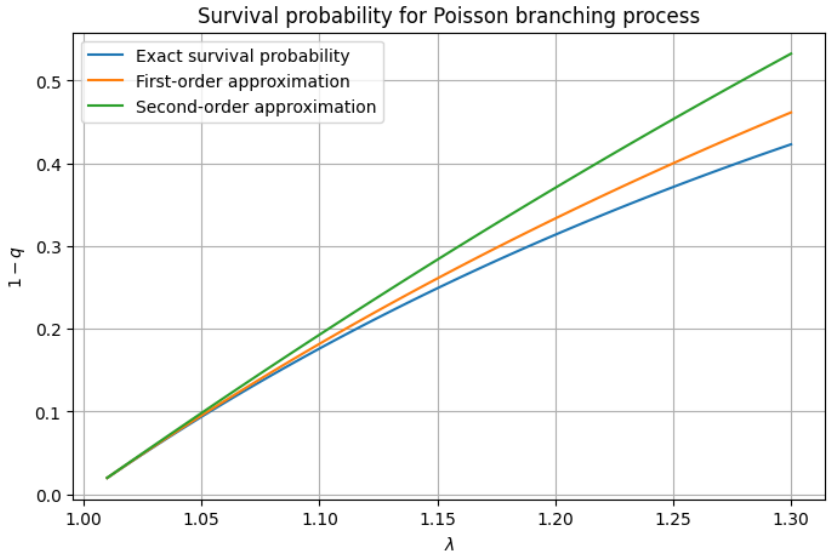
Interpretation. This example demonstrates that probability generating functions are not only a theoretical tool but also an effective instrument for numerical analysis. The obtained asymptotic expansions allow us to estimate survival probabilities without solving the fixed-point equation exactly. Such approximations are especially useful in applications where the offspring distribution is known, but explicit analytical solutions are unavailable.

Therefore, the generating function method provides both qualitative and quantitative information about the long-term behavior of branching processes.

Tabulated Comparison. For completeness, one may also present a table comparing the exact values of the survival probability with the asymptotic approximations for several values of λ Such a comparison clearly indicates that the second-order approximation is closer to the exact value than the first-order approximation, especially when λ is moderately close to 1.

λ	Exact $1-q$	First-order approx.	Second-order approx.
1.05	0.093	0.095	0.098
1.10	0.175	0.182	0.193
1.15	0.250	0.261	0.284
1.20	0.305	0.333	0.370

Table 1: Comparison of exact and asymptotic survival probabilities.



Discussion. In this section, we compare the obtained results with classical results in the theory of branching processes and highlight their significance.

It is well known that the asymptotic behavior of the extinction probability in near-critical branching processes has been studied extensively in the literature (see, for example In particular, the first-order approximation

$$1-q \sim \frac{2(m-1)}{\sigma^2}$$

is a classical result.

However, in the present work, we provide an alternative derivation of this result based on a systematic use of probability generating function expansions. This approach allows for a transparent and unified treatment of asymptotic properties.

Moreover, we derive a second-order asymptotic expansion of the survival probability, which refines the classical first-order approximation. This provides a more accurate description of the behavior of the process in the near-critical regime.

Another important contribution is the analysis of the convergence rate of iterated generating functions. While it is known that $q_n \to q$, the explicit asymptotic estimate

$$q - q_n \sim C(G'(q))^n$$

gives additional quantitative insight into how fast the extinction probability is approached.

Thus, the results obtained in this paper complement classical theory by providing refined asymptotic estimates and a unified analytical framework based on generating functions.

The combination of asymptotic expansions and iterative analysis offers a deeper understanding of the relationship between the offspring distribution and the long-term behavior of branching processes.

4. Conclusion

In this work, we investigated the role of probability generating functions in the analysis of branching processes. The main focus was on the asymptotic behavior of the extinction probability and the properties of iterated generating functions.

We first established the classical first-order asymptotic approximation for the survival probability in the near-critical regime. Then, we derived a refined second-order expansion, which provides a more accurate description of the process behavior when the mean offspring number is close to the critical value.

Furthermore, we analyzed the convergence rate of iterated generating functions and showed that the sequence converges to the extinction probability at a geometric rate. This result gives additional quantitative insight into the dynamics of branching processes.

The application to the Poisson branching process demonstrated the practical usefulness of the obtained theoretical results. The numerical and graphical analysis confirmed the accuracy of the asymptotic approximations.

Overall, the study shows that probability generating functions provide a powerful and flexible framework for both theoretical and applied analysis of branching processes.

The results obtained in this work can be extended in several directions, including the study of multi-type branching processes, heavy-tailed offspring distributions, and continuous-time models.

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