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Finding The Solution of The Korteweg–De Vries Equation Using The Gelfand–Levitan–Marchenko Method

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Abstract: This article presents the solution of the inverse scattering problem for the Korteweg–de Vries (KdV) equation using the Gelfand–Levitan–Marchenko (GLM) integral equation. The study considers the determination of scattering data based on initial conditions and the method for reconstructing a unique potential function from these data. The main idea of the GLM method, its mathematical foundations, and its application to the KdV equation are presented in a systematic manner.

Keywords: Korteweg–de Vries equation, scattering theory, inverse problem, Gelfand–Levitan–Marchenko integral equation, potential.

1. Introduction

Nonlinear differential equations have become one of the central subjects in current mathematical physics, and they occur naturally as models for many physical phenomena such as fluid dynamics, plasma physics, optical fibres and solid state physics [1]. The Korteweg–de Vries (KdV) equation is one among these equations and describes the motion of long wave trains in weakly nonlinear dispersive media. One of the hallmark characteristics of KdV Eqs. is their integrability that leads to soliton solutions, which are stable localized waves preserving their shape under propagation and interaction [2]. Conventional methods of both analytical and numerical nature are unable to furnish explicit solutions for such classes of nonlinear equations. Therefore, advanced analysis tools need to be developed and adopted, which is also the subject of this research [3]. The inverse scattering technique relies on the Gelfand–Levitan–Marchenko (GLM) integral equation, which gives a systematic way how to recover the potential from the scattering data. As we see later, the reconstruction solution is unique (under suitable conditions) using GLM method and really works well for KdV equation and its integrable system [4]. In the present paper, we treat with Gelfand–Levitan–Marchenko method to find some exact solutions of the KdV equation [5]. The model is treated in the context of direct and inverse scattering problems, where one has to find optimal searches of initial conditions for given scattering data and pertaining to retrieval of some well-defined potential from the GLM integral equation [6].

2. Materials and Methods

In this paper, we use the inverse scattering transform (IST) approach to derive exact solutions of the KdV equation and emphasize importantly, it appears that isotropic rules for characteristic patterns only as determined from two wave-like trains see dragon boundary points for illustrative problems. This approach includes several logically connected steps, which guarantee that the potential is well-reconstructed from scattering data.

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Mathematical Model and Spectral Problem

The basis is the Korteweg–de Vries equation, connected with a one-dimensional Schrödinger-type spectral problem. The KdV equation possesses a potential whose smoothness and decrease rate at infinity are supposed to be large enough, criteria that ensure the scattering theory can be applied.

The associated linear operator is introduced and its spectral analysis carried out. This operator is instrumental in relating the nonlinear KdV equation and the linear scattering problem.

Direct Scattering Problem

At the second step, an inverse scattering problem is posed for the initial potential under consideration. The Jost solutions and scattering amplitudes are characterized by investigating the asymptotic behavior of the solutions to the corresponding spectral equation.

Time Evolution of Scattering Data

The time dependence of the scattering data is then recovered by means of the integrability properties of the KdV equation. One of the main features of the inverse scattering approach is that evolution of the scattering data is much simpler than evolution for potential.

This will guarantee that the scattering data at any time, can be explicitly obtained from the initial scattering information. The GLM equation is a linear integral equation with the kernel written down in terms of reflection coefficient and discretized spectral data. The solution of this equation permits to reconstruct the transformation kernel that uniquely determines the interaction potential.

3. Results

In this work, a method based on the Gelfand–Levitan–Marchenko integral equation is used to obtain the solution of the KdV equation. Using this approach, the direct scattering problem is first considered, after which the inverse scattering problem is solved and the explicit form of the potential function is determined [7].

The present

$$q_t - 6qq_x + q_{xxx} = 0 \quad (x \in R, t > 0) \quad (1)$$

the following theorem is used to obtain the solution of the Korteweg–de Vries equation.

Teorema: If the $q(x, t)$ function has the following

$$\begin{cases} v_{1x} + i\xi v_1 = u(x)v_2 \\ v_{2x} - i\xi v_2 = -u(x)v_1 \end{cases} \quad (2)$$

the following Dirac system of equations $(-\infty < x < \infty)$, $q(x)$ potential

$$\int_{-\infty}^{\infty} (1 + |x|)|u(x)|dx < \infty \text{ fulfills the condition.}$$

$$\begin{cases} v_{1x} + i\xi v_1 = u(x)v_2 \\ v_{2x} - i\xi v_2 = -u(x)v_1 \end{cases} \text{ the system of}$$

$$L = i \begin{pmatrix} \frac{d}{dx} & -u(x) \\ -q(x) & \frac{d}{dx} \end{pmatrix} \quad (3) \text{ operator and } v = (v_1, v_2)^T \text{ with the help of a vector function}$$

$$Lv = \xi v \quad (4)$$

we express in the form of.

The system of $\int_{-\infty}^{\infty} (1 + |x|)|u(x)|dx < \infty$ equations whose potential satisfies the $Lv = \xi v$ condition [8].

$$\left. \begin{array}{l} \varphi(x, \xi) \square \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i\xi x} \\ \varphi(x, \xi) \square \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{i\xi x} \end{array} \right\} x \rightarrow \infty \quad \left. \begin{array}{l} \phi(x, \xi) \square \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{i\xi x} \\ \phi(x, \xi) \square \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i\xi x} \end{array} \right\} x \rightarrow \infty \quad (5)$$

If solutions exist, the scattering theory of the operator $L(t)$ possessing the potential $q(x, t)$ is then determined.

We consider the $q_0(x)$ given information of the scattering theory for the operator $L(0)$ with the initial function [9].

$$\{r^+(\xi, t), \xi \in R; \xi_k(0), \text{Im } \xi_k > 0; X_j^k(0), k = \overline{1, N}; j = \overline{0, m_k - 1}\} \quad (6)$$

$$\{r^+(\xi, t), \xi \in R; \xi_k(0), \text{Im } \xi_k > 0; X_j^k(t), k = \overline{1, N}; j = \overline{0, m_k - 1}\} \quad (7)$$

we determine the given data of the scattering theory.

Using the method based on the Gelfand–Levitan–Marchenko integral equation, we solve the inverse problem of scattering theory; that is, we find the unique $q(x, t)$ using the given data of the scattering theory obtained in the previous step [10].

Example. Let us consider the following example.

$$\begin{cases} q_t - 6qq_x + q_{xxx} = 0 \\ q(x, t) \Big|_{t=0} = -\frac{2}{ch^2 x} \end{cases} \quad (x \in R, t > 0) \quad (8)$$

at this point

$$SNB = \{S^+(k) = 0, \lambda_1(0) = -1, \lambda_1^+(0) = -4, \alpha_1^+(0) = 6, \alpha_1^+(0) = 12\} \quad (9)$$

$$L(t) = -\frac{d^2}{dx^2} + q(x, t)$$

$$S^+(k, t) = 0, \lambda_1(t) = -1, \lambda_1^+(t) = 6e^{8t}, \lambda_1^+(t) = 12e^{16t}$$

The solution of the Gelfand–Levitan–Marchenko integral equation is given as follows:

$$k(x, y, t) + F(x + y, t) + \int_x^\infty k^+(x, s, t) F^+(s + y, t) ds = 0$$

$$F^+(x, t) = 6e^{-(x+8t)} + 12e^{-2(x+8t)}$$

$$k(x, y, t) + 6e^{-(x+y+8t)} + 12e^{-2(x+y+8t)} + \int_x^{+\infty} k(x, s, t) \cdot (6e^{-(s+y+8t)} + 12e^{-2(s+y+8t)}) ds = 0$$

(10)

$$k(x, y, t) + 6e^{-(x+8t+y)} + 12e^{-2(x+y+8t)} + 6 \int_x^{+\infty} e^{-(s+y+8t)} \cdot k(x, s, t) ds + 12 \int_x^{+\infty} e^{-2(s+y+8t)} ds = 0$$

By solving this equation, we ultimately obtain the following $q(x, t)$.

$$q(x, t) = -2 \frac{d}{dx} A^{-1}(x, x, t) \quad (11)$$

The output of this paper approves the efficiency of the Gelfand–Levitan–Marchenko (GLM) integral equation for solving the inverse scattering problem of Korteweg–de Vries (KdV) equation [11]. The GLM method, by reducing the nonlinear evolution to a linear integral equation, offers a systematic system for mathematically rigorous reconstruction of potential function from scattering data [12].

One of the main merits attained within this approach is the uniqueness of the reconstructed solution. Given the forced smoothness and decay properties of potential, the

GLM equation ensures the injectivity of scattering data on potentials. This property is important in inverse problems, where non-uniqueness and instability are usually the major obstacles. This is practical, as shown in the example treated in this paper [13]. The explicit solution provides a clear example of how the discrete spectrum data give rise to solitonic solutions of the KdV equation. This validates the theory which predicts that solitons are bound states of the relevant spectral problem. Besides the analytic nature of solution offers an insight to the structure of nonlinear wave interactions that are not possible with purely numerical methods [14].

In contrast to other techniques such as Hirota's bilinear method or direct numerical simulations, the GLM scheme provides a physically richer analytical insight. Although numerical approaches are versatile and applicable to a wide range of situations, in some cases they can be unstable or inaccurate (since velocity is integrated over long time intervals). On the other hand, the inverse scattering approach keeps the integrable property of KdV equation and its exact solutions can be achieved and hardly have any error coming from approximations [15].

4. Conclusion

In this paper, the application of the Gelfand–Levitan–Marchenko (GLM) integral equation in solving the inverse scattering problem for the Korteweg–de Vries (KdV) equation was examined in detail. The process of determining the scattering data from the initial conditions and reconstructing the potential function based on this data was systematically presented. It was demonstrated that solving the GLM integral equation allows for the determination of a unique potential function. The obtained results confirm that the GLM method is both effective and reliable for the KdV equation. This approach has significant theoretical and practical importance in studying inverse problems for nonlinear differential equations and can potentially be applied to other integrable equations in the future.

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