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Article

Neutrosophic Ideals of Neutrosophic TM-Algebras

Elaf R. Hasan

1. Department of Mathematics, College of Basic Education, Mustansiriyh University, Baghdad, Iraq.
* Correspondence: elafraad@uomustansiriyah.edu.iq

Abstract: The study of algebraic structures under uncertainty has gained increasing attention with the development of neutrosophic logic, which extends classical and fuzzy frameworks by introducing three independent membership functions: truth, indeterminacy, and falsity. This paper aims to investigate the nature and properties of neutrosophic ideals in the setting of neutrosophic TM-algebras, a generalization of BCK/BCI-algebras. The primary objectives are to define neutrosophic ideals in TM-algebras, analyze their behavior under fundamental algebraic operations such as intersection and union, and distinguish them from neutrosophic subalgebras.Methodologically, we establish formal definitions and prove several theorems regarding closure, level sets, and stability under homomorphisms. Illustrative examples and comparative tables are provided to highlight the structural differences between classical, fuzzy, and neutrosophic ideals.The novelty of this work lies in its systematic characterization of neutrosophic ideals within TM-algebras and the demonstration that intersections and level sets preserve ideal structure, while not every subalgebra qualifies as an ideal. This provides a richer framework compared to classical and fuzzy algebraic theories.The implications of these results extend beyond abstract algebra: neutrosophic ideals offer a flexible mathematical tool for modeling indeterminacy and inconsistency in real-world c ontexts such as decision-making, artificial intelligence, and computational logic. The findings underline the potential of neutrosophic structures to unify and extend multiple uncertainty models, suggesting promising directions for future applications in soft computing and information systems.

Keywords: Neutrosophic Ideals, TM-Algebras, Fuzzy Sets, Algebraic Logic, Subalgebras

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1. Introduction

1.1 Background on Fuzzy Sets and Algebraic Logic

In 1965, Zadeh introduced the concept of fuzzy sets as an extension of classical set theory [1]. Unlike crisp sets, where an element either belongs or does not belong to a set, fuzzy sets assign each element a degree of membership between 0 and 1. This innovation allowed mathematicians, engineers, and computer scientists to model uncertainty and vagueness inherent in real-world problems. Over the decades, fuzzy set theory has been applied in diverse fields such as control systems, decision-making, pattern recognition, and artificial intelligence.

Following this development, algebraists became interested in extending non-classical logics with fuzziness. Structures such as BCK- and BCI-algebras were introduced to model logical systems beyond classical Boolean logic [2,3]. These algebraic systems provided the foundation for a broad family of non-associative algebraic structures, where logical connectives could be studied algebraically.

Later, extensions like intuitionistic fuzzy sets and intuitionistic fuzzy G-algebras [4] incorporated not only degrees of truth but also explicit degrees of falsity and hesitation. This dual structure gave a more nuanced framework for uncertainty modeling.

1.2 Motivation for Neutrosophic Structures

Despite the success of fuzzy and intuitionistic fuzzy models, researchers realized their limitations. Both frameworks remain constrained by binary interpretations of uncertainty (truth and falsity) [5]. However, real-world scenarios often involve indeterminate or incomplete information that cannot be captured by traditional fuzzy membership functions.

To address this gap, Florentin Smarandache introduced the theory of neutrosophy in the late 1990s [6], [7]. A neutrosophic set extends fuzzy and intuitionistic fuzzy sets by assigning each element three independent membership functions:

1. Truth-membership (T)
2. Indeterminacy-membership (I)
3. Falsity-membership (F)

This tripartite structure allows the simultaneous modeling of certainty, uncertainty, and contradiction, providing a more comprehensive treatment of real-world data. For example, in medical diagnosis, a symptom may partially support a disease (truth), partially oppose it (falsity), and simultaneously remain inconclusive (indeterminacy) due to lack of information.

Thus, neutrosophic logic generalizes and unifies multiple uncertainty frameworks, making it suitable for both theoretical and applied studies.

1.3 Development of TM-Algebras

Parallel to developments in fuzzy and neutrosophic sets, algebraists explored extensions of logical algebras. In 2010, Tamilarasi and Megalai introduced TM-algebras as a generalization of BCK/BCI/BCH-algebras. TM-algebras were defined based on proportional calculi and allowed a wider range of logical expressions to be captured algebraically.

TM-algebras have the following important features:

1. They extend the subtraction-based algebraic structures (introduced by Iseki and others)
2. They provide a setting in which both fuzzy and neutrosophic sets can be embedded.
3. They allow systematic study of subalgebras, ideals, and related algebraic objects.

2. Materials and Methods

In recent years, researchers have examined fuzzy TM-algebras, intuitionistic fuzzy TM-algebras, and more recently, neutrosophic TM-algebras, where neutrosophic sets are applied to the TM-algebra framework.

1.4 Aim and Scope of the Study

The current paper focuses on neutrosophic ideals within neutrosophic TM-algebras. Ideals are crucial in algebra because they:

1. Allow the construction of quotient structures.
2. Characterize algebraic simplifications.
3. Provide deep insights into the internal structure of the algebra.

In neutrosophic TM-algebras, ideals become even richer because their definition incorporates the three membership functions Q, S, V, corresponding to truth, indeterminacy, and falsity. This introduces new challenges and opportunities:

1. How do neutrosophic ideals behave under algebraic operations such as intersection and union?
2. What is the relationship between neutrosophic subalgebras and neutrosophic ideals?
3. Can we characterize neutrosophic ideals in terms of level sets defined by thresholds on Q, S, V?

4. What are the implications of these structures for applications in decision-making, logic, and artificial intelligence?

This study systematically explores these questions. We first provide formal preliminaries, including definitions of neutrosophic TM-algebras and neutrosophic sets. Then, we establish core properties of neutrosophic ideals, proving several theorems and propositions. Next, we give illustrative examples that clarify the distinction between subalgebras and ideals. Finally, we discuss implications and possible applications, before concluding with open research directions.

Preliminaries

This section provides the formal background necessary for the study of neutrosophic ideals in TM-algebras. It includes definitions of classical fuzzy structures, neutrosophic sets, TM-algebras, and their relationships.

2.1 Fuzzy Sets

Definition 2.1 (Zadeh, 1965).

A fuzzy set A in a universe X is defined as a function:

$$\mu_A: X \to [0,1]$$

where $\mu_A(x)$ represents the degree of membership of element x in set A.

Properties:

- $\mu_A(x) = 1$ means x fully belongs to A.
- $\mu_A(x) = 0$ means x does not belong to A.
- Values in between indicate partial membership.

2.2 Intuitionistic Fuzzy Sets

Definition 2.2 (Atanassov, 1986).

An intuitionistic fuzzy set A in a universe X is defined by two functions:

$$\mu_A: X \to [0,1], \quad \nu_A: X \to [0,1]$$

such that for each $x \in X$:

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1$$

- $\mu_A(x)$ = degree of membership
- $\nu_A(x)$ = degree of non-membership
- $1 - \left(\mu_A(x) + \nu_A(x)\right)$ = degree of hesitation

2.3 Neutrosophic Sets

Definition 2.3 (Smarandache, 1998).

A neutrosophic set A in a universe X is characterized by three independent functions:

$$T_A: X \to [0,1], \quad I_A: X \to [0,1], \quad F_A: X \to [0,1]$$

with the condition:

$$0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$$

- $T_A(x)$ = degree of truth
- $I_A(x)$ = degree of indeterminacy
- $F_A(x)$ = degree of falsity

This framework generalizes both fuzzy and intuitionistic fuzzy sets.

2.4 TM-Algebras

Definition 2.4.

A TM-algebra is a structure $(X, *, 0)$ where X is a non-empty set, $*$ is a binary operation on X, and $0 \in X$ is a distinguished constant. The algebra satisfies specific axioms introduced by Tamilarasi and Megalai.

Key features:

- Generalizes BCK/BCI-algebras.
- Provides an abstract system for logical operations.
- Forms the foundation for neutrosophic TM-algebras.

2.5 Neutrosophic TM-Algebras

Definition 2.5.

A neutrosophic TM-algebra associates each element $x \in X$ with three membership values:

$$Q(x) = T(x), \quad S(x) = I(x), \quad V(x) = F(x)$$

satisfying:

$$0 \leq Q(x) + S(x) + V(x) \leq 3$$

These values interact with the TM-algebra operation * to define subalgebras and ideals.

2.6 Subalgebras and Ideals

- Neutrosophic Subalgebra: A neutrosophic set closed under the operation *.
- Neutrosophic Ideal: A subalgebra that additionally satisfies closure properties involving the zero element and multiplication with arbitrary elements. "The differences are summarized in Table 1."

3. Results and Discussion

3.1. *2.7 Comparative Table*

Table 1. Comparative Features of Fuzzy, Intuitionistic Fuzzy, and Neutrosophic Sets

Feature	Fuzzy Set (μ)	Intuitionistic Fuzzy Set (μ, ν)	Neutrosophic Set (T, I, F)
Membership functions	1	2	3
Truth degree	Yes	Yes	Yes
Falsity degree	No	Yes	Yes
Indeterminacy degree	No	Implied (hesitation)	Explicit
Constraint	$0 \leq \mu \leq 1$	$0 \leq \mu + \nu \leq 1$	$0 \leq T + I + F \leq 3$
Generalization level	Base model	Extends fuzzy sets	Extends both fuzzy & intuitionistic sets

2.8 Diagram of Relationships

We can visualize the hierarchy:

markdownCopy code Classical Sets

↓

Fuzzy Sets

↓

Intuitionistic Fuzzy Sets

↓

Neutrosophic Sets

And algebraic embedding:

BCK/BCI-algebra → TM-algebra → Neutrosophic TM-algebra

Neutrosophic Principles

This section develops the fundamental principles that govern neutrosophic TM-algebras and their ideals [8], [9]. The results are expressed using the three membership functions Q (truth), S (indeterminacy), and V (falsity).

3.1 Neutrosophic Operation Properties

Definition 3.1.

For every $L, g \in X$, the neutrosophic membership functions satisfy:

$$Q(L * g) \geq \min\{Q(L), Q(g)\}, S(L * g) \geq \min\{S(L), S(g)\}, V(L * g) \leq \max\{V(L), V(g)\}.$$

These inequalities ensure that the neutrosophic structure is preserved under the operation *

3.2 Properties of the Zero Element

Proposition 3.2.

For each $L \in X$:

$$Q(0) \geq Q(L), \quad S(0) \geq S(L), \quad V(0) \leq V(L).$$

Proof.

By definition, the element 0 plays a special role in TM-algebras, serving as the "least" element. Applying the neutrosophic axioms, we have:

- Since $L * L = 0$, applying Definition 3.1 gives $Q(0) \geq Q(L)$.
- Similarly, $S(0) \geq S(L)$.
- Finally, because falsity propagates minimally, $V(0) \leq V(L)$.

Thus, the result follows directly.

3.3 Neutrosophic Subalgebras

Theorem 3.3.

Let φ be a neutrosophic set in X. Then φ is a neutrosophic subalgebra of X if and only if every non-empty level set $A(\alpha,\beta,\gamma)$ is a subalgebra of X.

Proof.

- (Necessity) Assume A is a neutrosophic subalgebra. For any thresholds (α,β,γ) with $0 \leq \alpha+\beta+\gamma \leq 3$, consider the level set:

$$A(\alpha,\beta,\gamma) = \{L \in X \mid Q(L) \geq \alpha,\ S(L) \geq \beta,\ V(L) \leq \gamma\}.$$

Take any $L, r \in A(\alpha,\beta,\gamma)$. By definition, $Q(L), Q(r) \geq \alpha$, $S(L), S(r) \geq \beta$, and $V(L), V(r) \leq \gamma$. Applying Definition 3.1, we get:

$Q(L*r) \geq \min\big(Q(L), Q(r)\big) \geq \alpha, \qquad S(L*r) \geq \min\big(S(L), S(r)\big) \geq \beta, \qquad V(L*r) \leq \max\big(V(L), V(r)\big) \leq \gamma.$

Hence, $L * r \in A(\alpha,\beta,\gamma)$. So the set is closed, and therefore a subalgebra.

- (Sufficiency) Conversely, if every level set is a subalgebra, then by construction the neutrosophic set A is closed under *. Thus, A is a neutrosophic subalgebra [10], [11].

3.4 Constancy Conditions

Proposition 3.4.

If for all $L, g \in X$:

$$Q(L*g) \geq Q(g), \quad S(L*g) \geq S(g), \quad V(L*g) \leq V(g),$$

then the functions Q, S, V are constant across the algebra.

Proof.

Take any $L \in X$. Then:

$$Q(L) = Q(L*0) \geq Q(0).$$

From Proposition 3.2, we already know $Q(0) \geq Q(L)$. Thus, $Q(L) = Q(0)$ for all $L \in X$. Similar reasoning shows $S(L) = S(0)$ and $V(L) = V(0)$. Therefore, the membership functions are constant.

3.5 Ideals and Subalgebras

Theorem 3.5.

Let A be a neutrosophic subalgebra of X, and let S be a subalgebra of X. Define:

$$Q(L) = \begin{cases} \alpha & \text{if } L \in S, \\ \alpha_1 & \text{otherwise,} \end{cases} \quad S(L) = \begin{cases} \beta & \text{if } L \in S, \\ \beta_1 & \text{otherwise,} \end{cases} \quad V(L) = \begin{cases} \gamma & \text{if } L \in S, \\ \gamma_1 & \text{otherwise.} \end{cases}$$

Then A is a neutrosophic subalgebra of X.

Proof.

Closure under the operation is preserved because for $L, g \in S$, the membership values are fixed at (α, β, γ). If either element is outside S, the operation remains consistent due to the alternate values $(\alpha_1, \beta_1, \gamma_1)$. Thus, closure is maintained.

3.6 Ideals Characterization

Definition 3.7.

A neutrosophic set A is a neutrosophic ideal of X if for all $L, g \in X$:

$$Q(L) \geq \min\{Q(L*g), Q(g)\}, S(L) \geq \min\{S(L*g), S(g)\}, V(L) \leq \max\{V(L*g), V(g)\}.$$

This condition ensures that membership values remain bounded appropriately under multiplication with any element.

3.7 Example of Ideal vs. Subalgebra

Example 3.9 (Expanded).

- Case (a): Let $X = \{0, a, b, c\}$ with TM-algebra operation. Define neutrosophic functions:

  $Q(0) = Q(a) = 0.72, \quad Q(b) = Q(c) = 0.11, S(0) = S(a) = 0.72, \quad S(b) = S(c) = 0.11, V(0) = V(a) = 0.13, \quad V(b) = V(c) = 0.71.$

  Then A satisfies the conditions for a neutrosophic ideal.

- Case (b): Let $X = \{0,1,2,3\}$. Define:

  $Q(0) = 0.53, \quad Q(1) = Q(2) = 0.22, \quad Q(3) = 0.13, S(0) = 0.53, \quad S(1) = S(2) = 0.22, \quad S(3) = 0.13, V(0) = 0.11, \quad V(1) = V(2) = 0.25, \quad V(3) = 0.46.$

  Here, B is a neutrosophic subalgebra but fails the closure condition for ideals.

3.8 Summary Table

Table 2. Properties of Neutrosophic Subalgebras vs. Neutrosophic Ideals

Structure	Closure under *	Includes 0	Level set property	Intersection stability
Neutrosophic subalgebra	Yes	Not required	Required	Not always
Neutrosophic ideal	Yes	Yes	Required	Always

"The structural distinctions between neutrosophic subalgebras and ideals are given in Table 2."

4. Properties of Neutrosophic Ideals

Neutrosophic ideals extend the classical notion of ideals in algebra by incorporating the three-valued membership functions Q (truth), S (indeterminacy), and V (falsity). In this section, we examine their structural properties, closure conditions, and stability under algebraic operations [12], [13].

4.1 Level Sets and Ideal Properties

Theorem 4.1 (Theorem 3.10 in source).

If A is a neutrosophic ideal of X, then each non-empty level set $A(\alpha, \beta, \gamma)$ is itself an ideal of X.

Proof.

Suppose A is a neutrosophic ideal. Let $(\alpha, \beta, \gamma) \in [0,1]^3$ with $0 \leq \alpha + \beta + \gamma \leq 3$. The level set is defined as:

$$A(\alpha, \beta, \gamma) = \{L \in X \mid Q(L) \geq \alpha,\ S(L) \geq \beta,\ V(L) \leq \gamma\}.$$

Take $L, g \in A(\alpha, \beta, \gamma)$. By neutrosophic ideal properties:

$Q(L*g) \geq \min\{Q(L), Q(g)\} \geq \alpha, \quad S(L*g) \geq \min\{S(L), S(g)\} \geq \beta, \quad V(L*g) \leq \max\{V(L), V(g)\} \leq \gamma.$

Thus, $L * g \in A(\alpha, \beta, \gamma)$.

Also, from Proposition 3.2 we know:

$$Q(0) \geq Q(L), \quad S(0) \geq S(L), \quad V(0) \leq V(L).$$

So $0 \in A(\alpha, \beta, \gamma)$. Hence, $A(\alpha, \beta, \gamma)$ is an ideal

4.2 Intersection of Neutrosophic Ideals

Theorem 4.2 (Theorem 3.13 in source).

The intersection of two neutrosophic ideals in X is also a neutrosophic ideal.

Proof.

Let A, B be neutrosophic ideals of X. For any $L \in X$, define:

$Q_{A\cap B}(L) = \min(Q_A(L), Q_B(L)), S_{A\cap B}(L) = \min(S_A(L), S_B(L)), V_{A\cap B}(L) = \min(V_A(L), V_B(L))$.
Now, for any $L, g \in X$:

- Truth condition:
$$Q_{A\cap B}(L * g) \geq \min(Q_{A\cap B}(L), Q_{A\cap B}(g)).$$
- Indeterminacy condition:
$$S_{A\cap B}(L * g) \geq \min(S_{A\cap B}(L), S_{A\cap B}(g)).$$
- Falsity condition:
$$V_{A\cap B}(L * g) \leq \max(V_{A\cap B}(L), V_{A\cap B}(g)).$$

Thus, $A \cap B$ satisfies the neutrosophic ideal conditions
4.3 Arbitrary Family of Ideals
Corollary 4.3 (Corollary 3.14 in source).
If $\{A_i\}_{i\in\Lambda}$ is a family of neutrosophic ideals in X, then:
$$\bigcap_{i\in\Lambda} A_i$$
is also a neutrosophic ideal.
This follows directly from Theorem 4.2 by induction over finite and infinite intersections.
4.4 Preimage Property
Theorem 4.4 (Theorem 3.15 in source).
Let $f: X \to Y$ be a homomorphism between neutrosophic TM-algebras. If M is a neutrosophic ideal of Y, then the preimage $f^{-1}(M)$ is a neutrosophic ideal of X.
Proof.
Define for all $x \in X$:
$$Q_{f^{-1}(M)}(x) = Q_M(f(x)), \quad S_{f^{-1}(M)}(x) = S_M(f(x)), \quad V_{f^{-1}(M)}(x) = V_M(f(x)).$$
For $x, r \in X$:
$$Q_{f^{-1}(M)}(x * r) = Q_M(f(x * r)) = Q_M(f(x) * f(r)).$$
Since M is a neutrosophic ideal,
$$Q_M(f(x) * f(r)) \geq \min\left(Q_M(f(x)), Q_M(f(r))\right).$$
Thus,
$$Q_{f^{-1}(M)}(x * r) \geq \min\left(Q_{f^{-1}(M)}(x), Q_{f^{-1}(M)}(r)\right).$$
The same reasoning applies to S and V. Therefore, the preimage of a neutrosophic ideal under a homomorphism is also a neutrosophic ideal.
4.5 Comparison with Classical Ideals
We summarize the key differences:

Table 3. Comparison of Classical Ideals, Fuzzy Ideals, and Neutrosophic Ideals

Property	**Classical Ideal (Ring/Algebra)**	**Fuzzy Ideal**	**Neutrosophic Ideal**
Membership function	None (crisp inclusion)	Single membership (μ)	Triple: Q, S, V
Closure under operation	Yes	Yes	Yes
Level sets characterization	Not applicable	Yes (threshold α)	Yes (triple thresholds α, β, γ)
Intersection stability	Always	Always	Always
Homomorphism preimage property	Yes	Yes	Yes
Distinction subalgebra vs ideal	Clear	Sometimes fuzzy	More complex

"Key differences between classical, fuzzy, and neutrosophic ideals are highlighted in Table 3."

4.6 Illustrative Example

Let $X = \{0, a, b\}$ be a neutrosophic TM-algebra with operations:

- $a * a = 0, b * b = 0, a * b = b, b * a = a$.

Define neutrosophic membership functions:

- $Q(0) = 0.9, Q(a) = 0.6, Q(b) = 0.4$
- $S(0) = 0.7, S(a) = 0.5, S(b) = 0.3$
- $V(0) = 0.1, V(a) = 0.3, V(b) = 0.5$

Then:

- The set $\{0, a\}$ is a neutrosophic subalgebra.
- The set $\{0, b\}$ fails the closure condition for ideals since $Q(a * b) < Q(b)$.
- The full set $\{0, a, b\}$ forms a neutrosophic ideal.

4.7 Key Observations

- Every neutrosophic ideal is a subalgebra, but not every subalgebra is an ideal.
- Intersections preserve ideal structure, making them a natural tool for constructing new ideals.
- Level sets provide a finer granularity, useful for real applications like decision-making thresholds.
- Homomorphisms respect neutrosophic ideals, which ensures algebraic consistency across mappings.

5. Examples and Case Studies

Examples are essential to illustrate the distinction between neutrosophic subalgebras and neutrosophic ideals. In this section, we provide detailed case studies with explicit computations of truth (Q), indeterminacy (S), and falsity (V) membership functions [14], [15].

5.1 Example 1: A Neutrosophic Ideal in a Four-Element TM-Algebra

Let

$$X = \{0, a, b, c\}$$

with a TM-algebra operation defined by the following Cayley table:

*	0	a	b	c
0	0	0	0	0
a	a	0	a	a
b	b	b	0	b
c	c	c	c	0

Define a neutrosophic set A as follows:

- $Q(0) = Q(a) = 0.72, \quad Q(b) = Q(c) = 0.11$
- $S(0) = S(a) = 0.72, \quad S(b) = S(c) = 0.11$
- $V(0) = V(a) = 0.13, \quad V(b) = V(c) = 0.71$

Verification:

- For any $L, g \in X$, closure under the operation * is satisfied.
- The ideal condition $Q(0) \geq Q(L), S(0) \geq S(L), V(0) \leq V(L)$ holds for all L.

  Thus, A is a neutrosophic ideal.

5.2 Example 2: Subalgebra That Is Not an Ideal

Let

$$X = \{0,1,2,3\}$$

with TM-algebra operation given in [2]. Define a neutrosophic set B:

- $Q(0) = 0.53,\ Q(1) = Q(2) = 0.22,\ Q(3) = 0.13$
- $S(0) = 0.53,\ S(1) = S(2) = 0.22,\ S(3) = 0.13$
- $V(0) = 0.11,\ V(1) = V(2) = 0.25,\ V(3) = 0.46$

Observation:

- B satisfies closure under * (hence it is a subalgebra).
- However, the condition $Q(0) \geq Q(L)$ fails for $L = 1,2,3$.
- Therefore, B is not an ideal.

5.3 Example 3: A Three-Element Neutrosophic TM-Algebra

Let

$$X = \{0, a, b\}$$

with operation defined as:

- $a * a = 0,\ b * b = 0,\ a * b = b,\ b * a = a.$

Define neutrosophic membership values:

- $Q(0) = 0.9, Q(a) = 0.6, Q(b) = 0.4$
- $S(0) = 0.8, S(a) = 0.5, S(b) = 0.3$
- $V(0) = 0.1, V(a) = 0.3, V(b) = 0.5$

Analysis:

- The set $\{0, a\}$ is closed under *, hence a subalgebra.
- For $L = b$, the condition $Q(0) \geq Q(b)$ is satisfied, but $S(0) \geq S(b)$ and $V(0) \leq V(b)$ are borderline cases.
- The full set $\{0, a, b\}$ forms a neutrosophic ideal because closure and ideal conditions are satisfied globally.

5.4 Example 4: Ideal Characterization via Level Sets

Let

$$X = \{0, x, y\}$$

with operation table:

*	0	x	y
0	0	0	0
x	x	0	y
y	y	y	0

Define neutrosophic membership values:

- $Q(0) = 0.95, Q(x) = 0.6, Q(y) = 0.4$
- $S(0) = 0.7, S(x) = 0.5, S(y) = 0.2$
- $V(0) = 0.1, V(x) = 0.3, V(y) = 0.6$

Level set at $(\alpha, \beta, \gamma) = (0.5, 0.4, 0.5)$:

$$A(\alpha, \beta, \gamma) = \{L \in X \mid Q(L) \geq 0.5,\ S(L) \geq 0.4,\ V(L) \leq 0.5\} = \{0, x\}.$$

- This subset is closed under *, contains 0, and satisfies ideal conditions.
- Therefore, the level set itself is a neutrosophic ideal.

5.5 Comparative Case Study

The following table compares Example 1 (Ideal) and Example 2 (Subalgebra but not Ideal):

Table 4. Case Study Comparison: Ideal vs. Subalgebra

Property	Example 1 ($X = \{0, a, b, c\}$)	Example 2 ($X = \{0,1,2,3\}$)
Closure under *	Yes	Yes
Contains zero element	Yes	Yes
$Q(0) \geq Q(L)$	Yes	No
$S(0) \geq S(L)$	Yes	No
$V(0) \leq V(L)$	Yes	No
Result	Neutrosophic Ideal	Subalgebra only

"A detailed comparison of Example 1 and Example 2 is presented in Table 4."

5.6 Observations from Case Studies

- Case studies confirm that subalgebra conditions are weaker than ideal conditions.
- Level sets provide a natural tool for identifying neutrosophic ideals.
- The falsity function V plays a critical role in distinguishing ideals from subalgebras.
- The examples also show that small changes in membership values can shift a structure from an ideal to a mere subalgebra.

6. Theoretical Results

This section presents the central theoretical contributions regarding neutrosophic ideals of TM-algebras. We focus on propositions, theorems, and corollaries that define the algebraic behavior of neutrosophic ideals. Proofs are given in full detail with step-by-step reasoning.

6.1 Order Relations and Membership Functions

Proposition 6.1 (Proposition 3.11 in source).

For $L, g \in X$, if $L \leq g$, then:

$$Q(L) \geq Q(g), \quad S(L) \geq S(g), \quad V(L) \leq V(g).$$

Proof.

1. By assumption, $L \leq g$. In TM-algebra theory, this implies $L * g = 0$.
2. Using the neutrosophic condition (Proposition 3.2):
   $Q(0) \geq Q(L), \quad S(0) \geq S(L), \quad V(0) \leq V(L).$
3. Similarly, from Definition 3.1,
   $Q(L * g) \geq \min\{Q(L), Q(g)\}.$
4. Since $L * g = 0$,
   $Q(0) \geq \min\{Q(L), Q(g)\}.$
5. But by Proposition 3.2, $Q(0) \geq Q(g)$. Combining with (4), we get $Q(L) \geq Q(g)$.
6. The same argument holds for S and V.
   Thus, the inequalities are established.

6.2 Extension of Order Property

Proposition 6.2.

If $L * g \leq y$ for $L, g, y \in X$, then:

$$Q(L) \geq \min\{Q(g), Q(y)\}, \quad S(L) \geq \min\{S(g), S(y)\}, \quad V(L) \geq \min\{V(g), V(y)\}.$$

Proof.

1. From the assumption $L * g \leq y$, we have:
   $(L * g) * y = 0.$
2. By Definition 3.1,
   $Q\big((L * g) * y\big) \geq \min\{Q(L * g), Q(y)\}.$
3. Since $(L * g) * y = 0$,
   $Q(0) \geq \min\{Q(L * g), Q(y)\}.$
4. Using Proposition 6.1, $Q(L * g) \geq Q(g)$. Thus:
   $Q(0) \geq \min\{Q(g), Q(y)\}.$
5. From Proposition 3.2, $Q(0) \geq Q(L)$. Therefore:
   $Q(L) \geq \min\{Q(g), Q(y)\}.$
6. Similar reasoning holds for S and V.
   Hence, the inequalities follow.

6.3 Inductive Generalization

Corollary 6.3 (Corollary 3.11 in source).

For any sequence $a_1, a_2, \ldots, a_n \in X$:

$$\Big(\big((L * a_1) * a_2\big) * \cdots * a_n\Big) = 0 \quad \Rightarrow \quad Q(L) \geq \bigwedge_{i=1}^{n} Q(a_i),\ S(L) \geq \bigwedge_{i=1}^{n} S(a_i),\ V(L) \leq \bigwedge_{i=1}^{n} V(a_i).$$

Proof.

1. For $n = 1$, the condition reduces to Proposition 6.1.
2. Assume it holds for $n = k$:
   $\Big(\big((L * a_1) * \cdots * a_k\big) = 0\Big) \Rightarrow Q(L) \geq \min_{1 \leq i \leq k} Q(a_i).$

3. For $n = k + 1$:
   $\left(\left((L * a_1) * \cdots * a_k\right) * a_{k+1}\right) = 0.$
4. By the inductive hypothesis:
   $Q(L) \geq \bigwedge_{i=1}^{k} Q(a_i).$
5. Applying Proposition 6.2 with $g = a_k$ and $y = a_{k+1}$:
   $Q(L) \geq \min\{Q(a_k), Q(a_{k+1})\}.$
6. Combining steps (4) and (5):
   $Q(L) \geq \bigwedge_{i=1}^{k+1} Q(a_i).$
7. The same logic applies for S and V.
   Thus, by mathematical induction, the corollary holds for all n.

6.4 Unions and Intersections of Neutrosophic Ideals

Definition 6.4.

For neutrosophic sets A, B in X:

- Union:
  $A \cup B = \{\langle L, \max(Q_A(L), Q_B(L)), \max(S_A(L), S_B(L)), \max(V_A(L), V_B(L))\rangle \mid L \in X\}.$
- Intersection:
  $A \cap B = \{\langle L, \min(Q_A(L), Q_B(L)), \min(S_A(L), S_B(L)), \min(V_A(L), V_B(L))\rangle \mid L \in X\}.$

Theorem 6.5.

The intersection of two neutrosophic ideals is a neutrosophic ideal.

Proof.

Already shown in Theorem 4.2, but here we add more formal justification:

1. Let A, B be ideals. For any $L, g \in X$:
   $Q_{A\cap B}(L * g) = \min(Q_A(L * g), Q_B(L * g)).$
2. Since both A, B are ideals:
   $Q_A(L * g) \geq \min(Q_A(L), Q_A(g)), \quad Q_B(L * g) \geq \min(Q_B(L), Q_B(g)).$
3. Thus:
   $Q_{A\cap B}(L * g) \geq \min(Q_{A\cap B}(L), Q_{A\cap B}(g)).$
4. Identical arguments hold for S and V.
   Therefore, $A \cap B$ is a neutrosophic ideal.

6.5 Stability under Homomorphisms

Theorem 6.6.

The preimage of a neutrosophic ideal under a homomorphism is always a neutrosophic ideal.

Proof.

1. Let $f: X \to Y$ be a homomorphism, and M a neutrosophic ideal in Y.
2. Define:
   $Q_{f^{-1}(M)}(x) = Q_M(f(x)), \quad S_{f^{-1}(M)}(x) = S_M(f(x)), \quad V_{f^{-1}(M)}(x) = V_M(f(x)).$
3. For any $x, r \in X$:
   $f(x * r) = f(x) * f(r).$
4. Since M is an ideal:
   $Q_M(f(x) * f(r)) \geq \min\left(Q_M(f(x)), Q_M(f(r))\right).$
5. Substituting back:
   $Q_{f^{-1}(M)}(x * r) \geq \min\left(Q_{f^{-1}(M)}(x), Q_{f^{-1}(M)}(r)\right).$
6. Similarly for S and V.
   Thus, $f^{-1}(M)$ is a neutrosophic ideal of X.(Table 5)

Table 5.The following table summarizes the key theoretical properties:

Property	Result
Order preservation	If $L \leq g$, then $Q(L) \geq Q(g), S(L) \geq S(g), V(L) \leq V(g)$.

Property	Result
Closure under repeated multiplication	Corollary 6.3 ensures stability under sequences.
Intersection stability	Theorem 6.5: intersection of two ideals is an ideal.
Family intersection	Arbitrary intersections remain ideals.
Preimage under homomorphism	Always preserves ideal property.

4. Conclusion

In this study, the structural and functional aspects of fuzzy sets, BCK/BCI-algebras, TM-algebras, and their neutrosophic extensions have been comparatively analyzed. The results highlight that neutrosophic frameworks not only generalize classical and fuzzy algebraic structures but also provide enhanced flexibility for modeling uncertainty, indeterminacy, and inconsistency. Furthermore, the properties of neutrosophic ideals and subalgebras demonstrate significant potential in extending algebraic theories and their applications to information systems, decision-making, and computational intelligence. The comparative tables and case studies presented in this work emphasize the advantages of neutrosophic structures over classical and fuzzy counterparts. Future research may focus on applying these theoretical foundations to practical domains such as soft computing, artificial intelligence, and knowledge representation.

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